Exact solution of the partially directed compact lattice animal model

1987 J. Phys. A: Math. Gen. 20 L543

(http://iopscience.iop.org/0305-4470/20/8/011)

J. Phys. A: Math. Gen. 20 (1987) L543-L547. Printed in the UK

LETTER TO THE EDITOR

Exact solution of the partially directed compact lattice animal model

V Privman and G Forgacs

Department of Physics, Clarkson University, Potsdam, New York 13676, USA

Received 12 February 1987

Abstract. We calculate exactly the number generating function for partially directed compact lattice animals on the square lattice. For the fully directed model, we report results of new numerical studies.

It is generally accepted that the number c_N of distinct N-site animals (i.e. connected site clusters) grows according to

$$c_N \approx CN^{-\theta}\lambda^N \qquad \text{as } N \to \infty \tag{1}$$

where θ is a universal exponent characteristic of the universality class. The constants C and $\lambda > 1$ depend on the lattice, connectivity range, etc. For ordinary $d = 2$ directed animals $\theta = \frac{1}{2}$ (Dhar *et al* 1982, Nadal *et al* 1982). Compact directed animal models were considered by Derrida and Nadal (1984) and, recently, by Bhat *et al* (1986). The two models have quite different scaling properties. In the Derrida-Nadal case, the number of sites, N, is partitioned in all possible non-increasing combinations

$$N = n_1 + n_2 + \cdots + n_k \qquad 1 \leqslant k \leqslant N \tag{2}$$

where $n_i \leqslant n_{i-1}$. Each group of n_i sites are then positioned at the points of the square lattice having (X, Y) coordinates $(i-1, 0)$, $(i-1, 1)$, $\cdots$, $(i-1, n_i - 1)$. The resulting animal is compact. There are

$$c_N \approx CN^{-1}\mu^{\sqrt{N}} \tag{3}$$

distinct animals of this type, where μ and C are exactly calculable. Guttmann and Hirschhorn (1984) commented that this identification of the partition of integers problem with compact animals (which is, in fact, well known in number theory, see, e.g., Andrews (1971)) can be further extended to the outward spiralling self-avoiding walk model (see Joyce (1984) for a full account and references).

Since the quantity

$$s_N \equiv (kT/N) \ln c_N \tag{4}$$

measures the entropy per site, the Derrida-Nadal model is exceptional. The generic scaling behaviour (1) corresponds to $s_N \to \ln \lambda > 0$, for large N. Bhat *et al* (1986) considered a different compact animal model for which relation (1) applies with $\theta = 0$ and $\lambda = 2.661\,85(\pm 5)$. In this letter we report an exact solution for a related model and also results of some new numerical studies for the Bhat *et al* problem. We first describe the new exactly solvable model for which the connectivity rules are easier to define.

The ordinary partially directed lattice animals on the square lattice with lattice spacing 1 are defined as N-site clusters which: (1) contain the origin $(0, 0)$; (2) do not include the point $(0, -1)$; (3) all the remaining $N-1$ sites are reachable *from the origin* by a partially directed walk of nearest-neighbour steps between cluster sites, in the $+X$ and $\pm Y$ directions. The condition (2) is needed to avoid animals that differ only by overall translations. For *compact* animals, we add the condition (4) that all the cluster sites with the same X coordinate form a sequence of nearest neighbours (there is no restriction in the case of a single site at a given X). Thus the compactness is imposed at each 'time' level, i.e. at each non-negative integer value along the directed axis X. Similarly to the fully directed model of Bhat *et al* (1986) which will be described below, we now construct relations for generating functions. First, let us denote by $c_N(k)$ the number of distinct N-site animals having exactly k 'root' sites at $X=0$, and $Y=0, 1, \cdots, k-1$. Clearly, $c_N(k)$ vanishes for $N<k$, and $c_k(k)=1$. The generating function for these k-root animals is defined by

$$F_k(z)=\sum_{N=k}^{\infty} c_N(k)z^{N-k} \tag{5}$$

so that $F_k(0)=1$. The generating function for *all* animals is then represented as

$$G(z)\equiv 1+\sum_{N=1}^{\infty} c_N z^N = 1+\sum_{k=1}^{\infty} z^k F_k(z) \tag{6}$$

where the conventional 'zero-site' term, 1, was added to have $G(0)=1$.

For each $N\geqslant k+1$ animal with root of k sites at $X=0$, all the sites *not in the root* form an $(N-k)$-site animal with root of m sites at $X=1$. This m-root is however not pinned with its lowest Y site at $Y=0$, but instead can occur in $(k+m-1)$ different locations, as implied by the connectivity rules. Thus we have the following relation:

$$F_k(z)=1+\sum_{m=1}^{\infty}(k+m-1)z^m F_m(z) \tag{7}$$

where the first term accounts for the k-site k-root animal. Remarkably, these relations can be combined in the form

$$F_k(z)-2F_{k-1}(z)+F_{k-2}(z)=0 \qquad \text{for } k\geqslant 3. \tag{8}$$

Thus, the solution must take the form

$$F_k(z)=A(z)k+B(z) \tag{9}$$

for *all* k. When (9) is substituted in the first two relations (7), $k=1$ and 2, we obtain a system of two linear equations for $A(z)$ and $B(z)$. The algebra is rather tedious and is not reproduced here. The results are

$$F_k(z)=\frac{kz(1-z)^3+(1-3z+z^2)(1-z)^2}{1-5z+7z^2-4z^3} \tag{10}$$

$$G(z)=\frac{1-4z+4z^2-z^3-z^4}{1-5z+7z^2-4z^3} \tag{11}$$

or

$$G(z)-1=\frac{z(1-z)^3}{1-5z+7z^2-4z^3}. \tag{12}$$

The singularity of $G(z)$ nearest to the origin is a simple pole at $z_c = 1/\lambda < 1$. This implies that the form (1) holds, with

$$\theta \equiv 0 \tag{13}$$

and

$$\lambda \equiv \frac{12}{7+(6\sqrt{177}-71)^{1/3}-(6\sqrt{177}+71)^{1/3}} \tag{14}$$

or

$$\lambda = 3.205\,569\,430\,40 \cdots. \tag{15}$$

Note that the cubic roots in (14) are both positive. Relation (1) applies to each $c_N(k)$ as $N \to \infty$ for fixed k, with appropriate coefficient $C(k)$ (see (1)). For the *total* number of animals, c_N, we have

$$C \equiv \frac{(\lambda-1)^3}{11\lambda^2-23\lambda+20} = 0.180\,915\,501\,88 \cdots \tag{16}$$

in terms of which the individual $C(k)$ are given by

$$C(k) = (k+\kappa)C\lambda^{-k} \tag{17}$$

$$\kappa \equiv \frac{\lambda^2-3\lambda+1}{\lambda-1} = 0.752\,171\,778\,88 \cdots. \tag{18}$$

The identical, up to prefactor, scaling of differently rooted animals is typical for models with asymptotic behaviour as in (1). A counterexample is provided by the Derrida–Nadal model (1984), where one can easily obtain

$$c_N(k) \approx \frac{1}{k!(k-1)!} N^{k-1} \qquad \text{for } N \gg k. \tag{19}$$

We now turn to the fully directed compact lattice animal model of Bhat *et al* (1986). Since these authors did not specify the compactness rules in detail, we list the definition for completeness. Thus, we define the k-root as k square lattice sites at

$$(X, Y) = (0, k-1), (1, k-2), \cdots, (k-1, 0) \tag{20}$$

so that the 1-root is just the origin, the 2-root consists of the points $(0, 1)$ and $(1, 0)$, etc. An N-site k-root animal, for $N \geq k+1$, is *directed* if all the $N-k$ sites which are not in the root can be reached from at least one root site by a directed walk of $+X$, $+Y$ steps between cluster sites. The animal will be *compact* if all the sites having the same $(X+Y)$ value form a sequence of diagonal neighbours (next-nearest neighbours on the square lattice). Finally, the root itself is a unique k-site k-root animal. Partial generating functions F_k are defined as in (5). Bhat *et al* (1986) considered animals rooted at the origin, the number of which is generated by

$$G_1(z) \equiv 1 + zF_1(z). \tag{21}$$

We prefer to study numerically the full generating function $G(z)$ (as in (6)), see below.

The analogue of relations (7) for the fully directed model is (Bhat *et al* 1986)

$$F_k(z) = 1 + \sum_{m=1}^{k+1} (k-m+2) z^m F_m(z) \qquad k \geq 1. \tag{22}$$

For $k \geq 3$, we have

$$F_{k+1}(z) = z^{-(k+1)}[F_k(z) - 2F_{k-1}(z) + F_{k-2}(z)]. \tag{23}$$

No exact solution of this infinite set of equations is known. However, power series expansion of $F_1(z)$, $F_2(z), \cdots, F_p(z)$, accurate to $O(z^p)$ inclusive, can be generated from the closed set of linear relations obtained by keeping the first p equations (22) without the last, $z^{p+1}F_{p+1}$ term in the pth equation. Bhat *et al* (1986) calculated the power series for $G_1(z)$, of (21), to $O(z^{20})$.

We aim at a very accurate estimation of the growth constant λ in (1), in view of the possible future progress with analytic study of (22) and (23). We generated a longer, 35-term series expansion of the full generating function $G(z)$, listed in table 1. On the grounds of universality, we now assume that $\theta \equiv 0$, and form approximants

$$\lambda_N^{(0)} = c_N / c_{N-1}. \tag{24}$$

Furthermore, based on numerical indications and the fact that in the exactly solvable model the singularity in $G(z)$ was a simple, isolated pole, we assume the following convergence pattern:

$$\lambda_N^{(0)} = \lambda + a\,\mathrm{e}^{-\alpha N} + \cdots \tag{25}$$

where the undisplayed terms may be mixed exponential-oscillating or higher-order exponential. The leading correction is monotonic since $\lambda_N^{(0)}$ decrease monotonically (see table 2). Correction of $O(\mathrm{e}^{-\alpha N})$ can be cancelled (to $O(\mathrm{e}^{-2\alpha N})$) by the transformation

$$\lambda_N^{(j+1)} = \frac{\lambda_N^{(j)}\lambda_{N-2}^{(j)} - \lambda_{N-1}^{(j)2}}{\lambda_N^{(j)} - 2\lambda_{N-1}^{(j)} + \lambda_{N-2}^{(j)}}. \tag{26}$$

Table 1. The number, c_N, of $N \leq 35$-site, arbitrary rooted compact fully directed lattice animals.

N	c_N	N	c_N
1	1	19	71 306 245
2	3	20	189 810 947
3	9	21	505 255 580
4	26	22	1 344 927 296
5	73	23	3 580 018 455
6	201	24	9 529 520 049
7	546	25	25 366 257 668
8	1 472	26	67 521 417 616
9	3 948	27	179 732 485 845
10	10 558	28	478 422 439 948
11	28 181	29	1 273 492 712 326
12	75 137	30	3 389 856 899 453
13	200 197	31	9 023 317 821 125
14	533 197	32	24 018 790 671 114
15	1 419 761	33	63 934 609 411 023
16	3 779 921	34	170 184 848 922 458
17	10 062 713	35	453 007 893 466 403
18	26 787 191		

Table 2. Values of $\lambda_N^{(j)}$ for $j=0, 1, 2$ (rounded to the last digit shown).

N	$\lambda_N^{(0)}$	$\lambda_N^{(1)}$	$\lambda_N^{(2)}$
30	2.661 858 106 170	2.661 857 942 718	2.661 857 944 075
31	2.661 858 033 766	2.661 857 943 344	2.661 857 943 941
32	2.661 857 993 618	2.661 857 943 647	2.661 857 943 930
33	2.661 857 971 389	2.661 857 943 814	2.661 857 944 020
34	2.661 857 959 097	2.661 857 943 890	2.661 857 943 954
35	2.661 857 952 307	2.661 857 943 930	2.661 857 943 972

The first iteration improves the convergence significantly (table 2). The resulting sequence $\lambda_N^{(1)}$ is again monotonic suggesting

$$\lambda_N^{(1)} = \lambda + b\,\mathrm{e}^{-\beta N} + \cdots \tag{27}$$

with $\beta > \alpha$. The second iteration results in a non-monotonic sequence (table 2). The values of $\lambda_N^{(2)}$, as well as other numerical estimates not reported here, suggest

$$\lambda = 2.661\,857\,9440\,(\pm 2). \tag{28}$$

In summary, the main result of this letter is the exact solution of the partially directed compact lattice animal model. The numerical estimate (28) for the fully directed case may be useful in future analytical studies. Compact directed animals form a distinct universality class, with $\theta \equiv 0$.

Instructive discussions with Professors M L Glasser and L S Schulman are greatly appreciated. Partial support for this research has been provided by the US National Science Foundation under grant DMR-86-01208, and by the Donors of the Petroleum Research Fund, administered by the American Chemical Society, under grant ACS-PRF-18175-G6. This financial assistance is gratefully acknowledged.

References

Andrews G E 1971 *Number Theory* (Philadelphia: Saunders)
Bhat V K, Bhan H L and Singh Y 1986 *J. Phys. A: Math. Gen.* **19** 3261-5
Derrida B and Nadal J P 1984 *J. Physique Lett.* **45** L701-6
Dhar D, Phani M K and Barma M 1982 *J. Phys. A: Math. Gen.* **15** L279-84
Guttmann A J and Hirschhorn M 1984 *J. Phys. A: Math. Gen.* **17** 3613-4
Joyce G S 1984 *J. Phys. A: Math. Gen.* **17** L463-7
Nadal J P, Derrida B and Vannimenus J 1982 *J. Physique* **43** 1561-74